An introduction to fractional calculus

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Fractional calculus

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- The subject has received attention of many scientists in mathematics, physics and engineering.
- It has become a hot issue in recent years.

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- The reason for this is that many real-world physical systems display fractional order dynamics and their behavior is governed by fractional differential equations (FDEs).
- The most important advantage of using FDEs is their nonlocal property [1]. This means that the next state of a dynamical system depends not only on its current state but also on all of its previous states.
- Therefore, the memory effect of these derivatives is one of the main reasons to use them in various applications.

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- Therefore, approximate methods for finding the approximate solutions for such problem are very necessary and useful.

Some definitions of fractional calculus

Definition

The fractional integration operator of order $\alpha \geq 0$ of a function $f(t)$ in the Riemann-Liouville sense is defined as [1]:

$$(I^{\alpha}f)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, & \alpha > 0, \\ f(t), & \alpha = 0, \end{cases} \quad (1)$$

where $\Gamma()$ is the Gamma function and α is a positive constant.

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The fractional derivative operator of order $q-1<\alpha\leq q$ of a function $f(t)$ in the Riemann-Liouville sense is defined as [1]:

$$\left({}_0D_t^{\alpha}f\right)(t)=\begin{cases}\dfrac{1}{\Gamma(q-\alpha)}\dfrac{d^q}{dt^q}\displaystyle\int_0^t (t-\tau)^{q-\alpha-1}f(\tau)\,d\tau, \\ \qquad\qquad q-1<\alpha<q, \\ \dfrac{d^q f(t)}{dt^q}, \qquad \alpha=q,\end{cases} \tag{2}$$

where $q\in\mathbb{N}$.

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Definition

The fractional derivative operator of order $q-1<\alpha\leq q$ of a function $f(t)$ in the Caputo sense is defined as [1]:

$$\left({}_0^cD_t^\alpha f\right)(t)=\begin{cases}\dfrac{1}{\Gamma(q-\alpha)}\displaystyle\int_0^t(t-\tau)^{q-\alpha-1}\frac{d^qf(\tau)}{d\tau^q}\,d\tau, & \\ & q-1<\alpha<q, \\ \dfrac{d^qf(t)}{dt^q}, & \alpha=q,\end{cases}\tag{3}$$

where $q\in\mathbb{N}$.

Useful properties of fractional operators

Remark

Note that based on the definitions of the fractional integration in the Riemann-Liouville sense and derivative in the Caputo sense, we have the following useful properties [1]:

$$I^{\alpha} t^{m} = \frac{m!}{\Gamma(m+\alpha+1)}\, t^{m+\alpha}, \qquad m \in \mathbb{N}, \tag{4}$$

and

$${}_{0}^{c}D_{t}^{\alpha} t^{m} = \begin{cases} \dfrac{m!}{\Gamma(m-\alpha+1)}\, t^{m-\alpha}, & q \leq m \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

where $q-1 < \alpha \leq q$.

Useful properties of fractional operators

Remark

The useful relation between the Riemann-Liouvill operator and Caputo operator is given by the following expression [1]:

$$\left(I^{\alpha}\, {}_0^cD_t^{\alpha} f\right)(t) = f(t) - \sum_{k=0}^{q-1} f^{(k)}(0^+)\frac{t^k}{k!}, \quad t > 0, \tag{6}$$

$$\left({}_0^cD_t^{\alpha} I^{\alpha} f\right)(t) = f(t), \tag{7}$$

where $q - 1 < \alpha \leq q$ *and* $q \in \mathbb{N}$.

Fractional differential equations (FDEs)

A general form of FDE can be expressed as follows [2]:

$$ {}_0^cD_x^{\alpha_q}u(x) = g\left(x, u(x), {}_0^cD_x^{\alpha_1}u(x), {}_0^cD_x^{\alpha_2}u(x), \ldots, {}_0^cD_x^{\alpha_{q-1}}u(x)\right), $$

where $0 \le \alpha_i \le \alpha_q \le q$ for $i = 1, 2, \ldots, q-1$ and $q-1 < \alpha_q \le q$, subject to the initial conditions:

$$ u^{(j)}(0) = u_0^j, \qquad j = 0, 1, 2, \ldots, q-1. $$

in which $g : [0,1] \times \mathbb{R}^q \to \mathbb{R}$ is a given continuous mapping. It is assumed that $\sigma_i - 1 < \alpha_i \le \sigma_i$, where σ_i for $i = 1, 2, \ldots, q-1$ are positive integer constants.

A well-known FDEs

- Time fractional diffusion-wave equation with damping [3]:

$$ {}_0^cD_t^\alpha u(x,t) + u_t(x,t) = u_{xx}(x,t) + f(x,y), \;\; 1 < \alpha \leq 2, $$

subject to the initial and boundary conditions

$$ u(x,0) = g_0(x), \qquad u_t(x,0) = g_1(x), $$
$$ u(0,t) = h_0(t), \qquad u(1,y) = h_1(t), $$

where ${}_0^cD_t^\alpha$ denotes the fractional derivative of order $1 < \alpha \leq 2$ in the Caputo sense.

Variable-order fractional calculus

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Variable-order fractional calculus

- Since the order of fractional derivatives and integrals may take any arbitrary value, another extension is considering the order not to be constant [4, 5]. This provides an extension of the classical fractional calculus, namely variable-order fractional calculus.
- In fact this subject is a generalization of fractional calculus where the order of fractional derivatives are known functions which depends on the time.
- Recently, several researchers have investigated and shown that many complex physical models can be described via variable-order fractional derivatives with a great success.

Variable-order fractional calculus

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Variable-order fractional calculus

- Variable-order fractional derivatives are very useful when the memory properties change with time and spatial location.
- Sun et al. [6] have investigated the advantages of using variable-order fractional derivatives rather than constant order fractional derivatives.
- Analytically handling equations described by the variable-order fractional derivatives is extremely difficult, and even for most cases impossible due to their high complexity.
- So, presenting efficient numerical methods to find their numerical solutions is of great importance in practice.

Some definitions of varaible-order fractional calculus

Definition

The variable-order fractional integration operator of order $\alpha(t) \geq 0$ of a function $f(t)$ in the Riemann-Liouville sense is defined as [4, 5]:

$$\left(I^{\alpha(t)}f\right)(t) = \begin{cases} \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-\tau)^{\alpha(t)-1}f(\tau)d\tau, & \alpha(t) > 0, \\ f(t), & \alpha(t) = 0, \end{cases} \tag{8}$$

where $\Gamma()$ is the Gamma function and $\alpha(t)$ is a function with respect to variable t.

Some definitions of variable-order fractional calculus

Definition

The variable-order fractional derivative operator of order $q-1<\alpha(t)\leq q$ of a function $f(t)$ in the Caputo sense is defined as [4, 5]:

$$\left({}_{0}^{c}D_{t}^{\alpha(t)}f\right)(t)=\begin{cases}\dfrac{1}{\Gamma(q-\alpha(t))}\displaystyle\int_{0}^{t}(t-\tau)^{q-\alpha(t)-1}\frac{d^{q}f(\tau)}{d\tau^{q}}\,d\tau, & \\ & q-1<\alpha(t)<q, \\ \dfrac{d^{q}f(t)}{dt^{q}}, & \alpha(t)=q,\end{cases} \tag{9}$$

where $q\in\mathbb{N}$.

Useful properties of Varaible-order fractional operators

Remark

Note that based on the definitions of the variable-order fractional integration in the Riemann-Liouville sense and derivative in the Caputo sense, we have the following useful properties [4, 5]:

$$I^{\alpha(t)} t^m = \frac{m!}{\Gamma(m+\alpha(t)+1)} t^{m+\alpha(t)}, \qquad m \in \mathbb{N}, \tag{10}$$

and

$$ {}_0^c D_t^{\alpha(t)} t^m = \begin{cases} \dfrac{m!}{\Gamma(m-\alpha(t)+1)} t^{m-\alpha(t)}, & q \leq m \in \mathbb{N}, \\ 0, & \textit{otherwise}, \end{cases} \tag{11}$$

where $q-1 < \alpha(t) \leq q$.

Variable-order fractional differential equations (V-OFDEs)

A general form of the V-OFDEs can be defined as follows:

$$ {}_0^cD_t^{\alpha(t)}u(t) = f(t, u(t), {}_0^cD_t^{\alpha_1(t)}u(t), {}_0^cD_t^{\alpha_2(t)}u(t), \ldots, {}_0^cD_t^{\alpha_n(t)}u(t)), $$

subject to the initial conditions

$$ u^{(i)}(0) = u_0^{(i)}, \qquad i = 0, 1, ..., q-1, $$

where q is the integer such that $q-1 < \alpha(t) \leq q$, $0 < \alpha_1(t) < \alpha_2(t) < \ldots < \alpha_n(t) < \alpha(t)$. Also, $D_t^{\alpha(t)}u(t)$ denotes the variable-order fractional derivative of order $\alpha(t)$ in the Caputo type for $u(t)$.

Numerical example

Example

Consider the following V-FDE:

$$
\begin{aligned}
{}_0^cD_t^{\alpha(t)}u(t) + \sin t\ {}_0^cD_t^{\beta(t)}u(t) + \cos t\ u(t) \\
= \frac{6t^{3-\alpha(t)}}{\Gamma(4-\alpha(t))} + \frac{6\sin t\ t^{3-\beta(t)}}{\Gamma(4-\beta(t))} + t^3\cos t,
\end{aligned}
$$

where the initial conditions are $u(0) = u^{'}(0) = 0$, and $1 < \alpha(t) \leq 2,\ 0 < \beta(t) \leq 1.$ The exact solution for this problem is $u(t) = t^3$. This problem can be solved by considering $\alpha(t) = 2 - \sin^2(t)$ and $\beta(t) = 1 - \frac{e^{-t^3}}{6}$.

Some well-known V-OFDEs

- Variable-order fractional Poisson equation [7]:

$$ {}^{c}_{0}D_x^{\alpha(x,y)}u(x,y) + {}^{c}_{0}D_y^{\beta(x,y)}u(x,y) = f(x,y), $$

subject to the Dirichlet boundary conditions

$$ u(x,0) = g_0(x), \qquad u(0,y) = h_0(y), $$
$$ u(x,1) = g_1(x), \qquad u(1,y) = h_1(y), $$

where ${}^{c}_{0}D_x^{\alpha(x,y)}$ and ${}^{c}_{0}D_y^{\beta(x,y)}$ denote the fractional derivatives of orders $1 < \alpha(x,y) \leq 2$ and $1 < \beta(x,y) \leq 2$, respectively.

Some well-known V-OFDEs

- Variable-order space-time fractional telegraph equation [8]:

$$ {}_{0}^{c}D_t^{\alpha(x,t)} u(x,t) + 2\rho\, {}_{0}^{c}D_t^{\alpha(x,t)-1} u(x,t) + \sigma^2 u(x,t) = {}_{0}^{c}D_x^{\beta(x,t)} u(x,t) + f(x,t), $$

subject to the initial and boundary conditions

$$ u(x,0) = g_0(x), \qquad u(0,t) = h_0(t), $$
$$ u_t(x,0) = g_1(x), \qquad u(1,t) = h_1(t). $$

Here, ${}_{0}^{c}D_t^{\alpha(x,t)}$ and ${}_{0}^{c}D_t^{\alpha(x,t)-1}$ denote the variable-order fractional derivatives of orders $1 < \alpha(x,t) \leq 2$ and $0 < \alpha(x,t) - 1 \leq 1$ with respect to the time variable t, respectively and ${}_{0}^{c}D_x^{\beta(x,t)}$ denotes the variable-order fractional derivative of order $1 < \beta(x,t) \leq 2$ with respect to space variable x.

Some well-known V-OFDEs

- Variable-order fractional biharmonic equation [9]:

$$
{}_0^cD_x^{2\alpha(x,y)}u(x,y)+2\,{}_0^cD_y^{\beta(x,y)}\left({}_0^cD_x^{\alpha(x,y)}u(x,y)\right) + {}_0^cD_y^{2\beta(x,y)}u(x,y) = f(x,y),
$$

subject to the boundary conditions:

$$
\begin{aligned}
u(x,0) &= g_1(x), & u(0,y) &= g_3(y),\\
u(x,1) &= g_2(x), & u(1,y) &= g_4(y),\\
u_y(x,0) &= h_1(x), & u_x(0,y) &= h_3(y),\\
u_y(x,1) &= h_2(x), & u_x(1,y) &= h_4(y).
\end{aligned}
$$

${}_0^cD_x^{\alpha(x,y)}$ and ${}_0^cD_y^{\beta(x,y)}$ denote the fractional derivatives of orders $1 < \alpha(x,y) \leq 2$ and $1 < \beta(x,y) \leq 2$, respectively.

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